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| **Introduction**  In many problems, a function $ f(t), \;\; t \in [a, \; b]$ is transformed to another function $ F(s)$ through a relation of the type:  $\displaystyle F(s) = \int_a^b K(t,s) f(t) dt$  where $ K(t,s)$ is a known function. Here, $ F(s)$ is called integral transform of $ f(t)$ . Thus, an integral transform sends a given function $ f(t)$ into another function $ F(s)$ . This transformation of $ f(t)$ into $ F(s)$ provides a method to tackle a problem more readily. In some cases, it affords solutions to otherwise difficult problems. In view of this, the integral transforms find numerous applications in engineering problems. Laplace transform is a particular case of integral transform (where $ f(t)$ is defined on $ [0, \infty)$and $ K(s,t) = e^{-st}$ ). As we will see in the following, application of Laplace transform reduces a linear differential equation with constant coefficients to an algebraic equation, which can be solved by algebraic methods. Thus, it provides a powerful tool to solve differential equations.  It is important to note here that there is some sort of analogy with what we had learnt during the study of logarithms in school. That is, to multiply two numbers, we first calculate their logarithms, add them and then use the table of antilogarithm to get back the original product. In a similar way, we first transform the problem that was posed as a function of $ f(t)$ to a problem in $ F(s)$ , make some calculations and then use the table of inverse Laplace transform to get the solution of the actual problem. |
| |  | | --- | | **Table 10.1:** Laplace transform of some Elementary Functions | | |  |  |  |  | | --- | --- | --- | --- | | $ f(t)$ | $ {\mathcal L}(f(t))$ | $ f(t)$ | $ {\mathcal L}(f(t))$ | |  |  |  |  | | 1 | $ \displaystyle \frac{1}{s},\;\; s> 0$ | $ t$ | $ \displaystyle\frac{1}{s^2},\;\; s> 0$ | |  |  |  |  | | $ t^n$ | $ \displaystyle \frac{n!}{s^{n+1}},\;\; s> 0$ | $ e^{at} $ | $ \displaystyle \frac{1}{s-a},\;\; s> a$ | |  |  |  |  | | $ \sin(at)$ | $ \displaystyle\frac{a}{s^2 + a^2},\;\; s> 0$ | $ \cos(at)$ | $ \displaystyle\frac{s}{s^2 + a^2},\;\; s> 0$ | |  |  |  |  | | $ \sinh(at)$ | $ \displaystyle \frac{a}{s^2 - a^2},\;\; s> a$ | $ \cosh(at)$ | $ \displaystyle\frac{s}{s^2 - a^2},\;\; s> a$ | |  Properties of Laplace Transform **LEMMA 10.3.1 (Linearity of Laplace Transform)**   1. Let $ a, b \in {\mathbb{R}}$ . Then  |  |  |  |  | | --- | --- | --- | --- | | $\displaystyle {\mathcal L}\bigl(a f(t) + b g(t)\bigr)$ | $\displaystyle =$ | $\displaystyle \int_0^\infty \bigl(a f(t) + b g(t)\bigr) e^{-s t} dt$ |  | |  | $\displaystyle =$ | $\displaystyle a {\mathcal L}(f(t)) + b {\mathcal L}(g(t)).$ |  |  1. If $ F(s) = {\mathcal L}(f(t)), \; $ and $ G(s) = {\mathcal L}(g(t))$ , then   $\displaystyle {\mathcal L}^{-1} \bigl( a F(s) + b G(s) \bigr) = a f(t) + b g(t).$  **THEOREM 10.3.3 (Scaling by $ a$ )   *Let $ f(t)$ be a piecewise continuous function with Laplace transform $ F(s).$ Then for$ \; a > 0,\;\; {\mathcal L}(f(at)) = \displaystyle\frac{1}{a} F(\frac{s}{a}).$***  **THEOREM 10.3.5 (Laplace Transform of Differentiable Functions)*****Let $ f(t),$ for $ t > 0,$ be a differentiable function with the derivative, $ f^\prime(t),$ being continuous. Suppose that there exist constants $ M$ and $ T$ such that $ \vert f(t) \vert \leq M e^{{\alpha}t}$ for all $ t \geq T.$ If $ {\mathcal L}(f(t)) = F(s)$ then***   |  |  | | --- | --- | | $\displaystyle {\mathcal L}\left( f^\prime(t) \right) = s F(s) - f(0) \;\; {\mbox{ for }} \;\; s > {\alpha}.$ | (10.3.1) |   **COROLLARY 10.3.6   *Let $ f(t)$ be a function with $ {\mathcal L}(f(t)) = F(s).$ If $ f^\prime(t), \ldots, f^{(n-1)}(t), f^{(n)}(t)$ exist and $ f^{(n)}(t)$ is continuous for $ t \geq 0,$ then***   |  |  | | --- | --- | | $\displaystyle {\mathcal L}\bigl( f^{(n)}(t)\bigr) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^\prime(0) - \cdots - f^{(n-1)}(0).$ | (10.3.2) |     ***In particular, for $ n = 2$ , we have***   |  |  | | --- | --- | | $\displaystyle {\mathcal L}\bigl( f^{\prime\prime}(t) \bigr) = s^2 F(s) - s f(0) - f^\prime(0).$ | (10.3.3) |     **COROLLARY 10.3.7   *Let $ f^\prime(t)$ be a piecewise continuous function for $ t \geq 0$ . Also, let $ f(0) = 0$ . Then***  $\displaystyle {\mathcal L}(f^\prime(t))= s F(s) \;\; {\mbox{ or equivalently }} \;\; {\mathcal L}^{-1}(s F(s)) = f^\prime(t).$  **LEMMA 10.3.9 (Laplace Transform of $ t f(t)$ )   *Let $ f(t)$ be a piecewise continuous function with $ {\mathcal L}(f(t)) = F(s).$ If the function $ F(s)$ is differentiable, then***  $\displaystyle {\mathcal L}(t f(t)) = -\displaystyle\frac{d}{ds}F(s).$  $\displaystyle {\mbox{ Equivalently,}} \;\;\; {\mathcal L}^{-1}(-\frac{d}{ds}F(s)) = t f(t).$  **COROLLARY 10.3.10   *Let $ {\mathcal L}(f(t)) = F(s)$ and $ g(t) = \displaystyle\frac{f(t)}{t}.$ Then***  $\displaystyle {\mathcal L}(g(t)) = G(s) = \int\limits_s^\infty F(p) dp.$  **LEMMA 10.3.12 (Laplace Transform of an Integral)*****If $ F(s) = {\mathcal L}(f(t))$ then***  $\displaystyle {\mathcal L}\left[\int_0^t f(\tau) d\tau \right] = \frac{F(s)}{s}.$  ***Equivalently, $ {\mathcal L}^{-1} \left( \displaystyle\frac{F(s)}{s} \right) = \int_0^t f(\tau) d \tau.$***  **LEMMA 10.3.14 ($ s$ -Shifting)*****Let $ {\mathcal L}(f(t)) = F(s).$ Then $ {\mathcal L}(e^{at} f(t)) = F(s-a)$ for $ s > a.$*** Limiting Theorems The following two theorems give us the behaviour of the function $ f(t)$ when $ t \longrightarrow 0^+$ and when $ t \longrightarrow \infty$ .  **THEOREM 10.4.1 (First Limit Theorem)*****Suppose $ {\mathcal L}(f(t))$ exists. Then***  $\displaystyle \lim_{t \longrightarrow 0^+} f(t) = \lim_{s \longrightarrow \infty} s F(s).$  **THEOREM 10.4.3 (Second Limit Theorem)*****Suppose $ {\mathcal L}(f(t))$ exists. Then***  $\displaystyle \lim_{t \longrightarrow \infty} f(t) = \lim_{s \longrightarrow 0} s F(s)$  ***provided that $ s F(s)$ converges to a finite limit as $ s$ tends to 0 .***  **DEFINITION 10.4.5 (Convolution of Functions)   *Let $ f(t)$ and $ g(t)$ be two smooth functions. The convolution, $ f \star g,$ is a function defined by***  $\displaystyle (f\star g) (t) = \int_0^t f(\tau) g( t - \tau) d \tau.$  **THEOREM 10.4.6 (Convolution Theorem)*****If $ F(s) = {\mathcal L}(f(t))$ and $ G(s) = {\mathcal L}(g(t))$ then***  $\displaystyle {\mathcal L}\left[\int_0^t f(\tau) g(t - \tau) d\tau \right] = F(s) \cdot G(s).$ |